Business PreCalculus MATH 1643 Section 004, Spring 2014
Lesson 18: Inverse Functions

Definition 1. One-to-One Function: *A function is called a* ***one-to-one*** *if each y-value in its range corresponds to only one x-value in its domain.*

The geometric interpretation of a one-to-one function is called the horizontal line test.

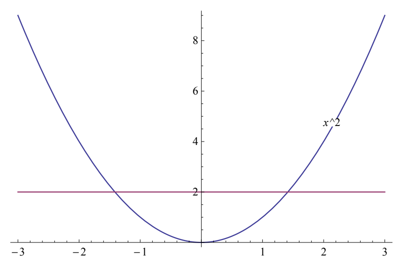
Definition 2. Horizontal Line Test: *A function f is one-to-one if* **no** *horizontal line intersects the graph of f in more than one point.*

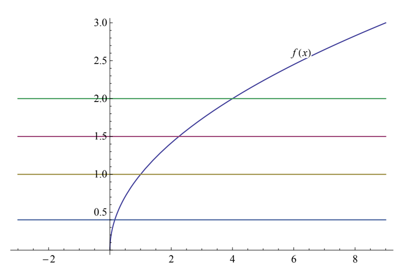
Example 1. *Use the horizontal line test to determine which of the following functions are one-to-one.*

a. $f(x) = \sqrt{x}$.

b. $h(x) = x^2$.

Solution: *To decide whether the function is one-to-one or not using the horizontal line test, we need to sketch its graph.*





We see that no horizontal line intersects the graph f in more than one point, then $f(x) = \sqrt{x}$ is one-to-one. The function $h(x) = x^2$ is not one-to-one because the horizontal line $y = 2$ intersects the graph of h in two points.

Definition 3. Inverse Function: *Let $f(x)$ be a* ***one-to-one function****. Then we define the* ***inverse of*** *f, denoted by f^{-1},*

$$f^{-1}(y) = x \text{ if and only if } y = f(x)$$

Remark 1. *From the definition of the inverse function, we have*

$$\text{Domain of } f = \text{Range of } f^{-1} \text{ and Domain of } f^{-1} = \text{Range of } f$$

Remark 2. <u>§§***WARNING***§§</u> *The notation* $f^{-1}(x)$ ***DOES NOT*** *mean* $\frac{1}{f(x)}$.

Example 2. *Let* $f(3) = 5$, *then find* $f^{-1}(5)$.

<u>***Solution:***</u> *By the definition* $f^{-1}(y) = x$ *if and only if* $y = f(x)$, *then letting* $x = 3$ *and* $y = 5$ *implies* $5 = f(3)$ *if and only if* $f^{-1}(5) = 3$. *So,* $f^{-1}(5) = 3$.

Definition 4. <u>**Inverse Function Property:**</u> *Let* $f(x)$ *be a* ***one-to-one function****. Then*

1. $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ *for every* x *in the domain of* f^{-1}.
2. $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ *for every* x *in the domain of* f.

This means that f^{-1} *undoes anything that* f *does to* x.

Example 3. *Verify that the following pairs of functions are inverses of each other.*

$$f(x) = 2x + 3 \text{ and } g(x) = \frac{x-3}{2}.$$

<u>***Solution:***</u>

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\frac{x-3}{2}) \\ &= 2(\frac{x-3}{2}) + 3 \\ &= x\end{aligned}$$

So $(f \circ g)(x) = x$. *Moreover,*

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x+3) \\ &= \frac{(2x+3)-3}{2} \\ &= x\end{aligned}$$

Then $(g \circ f)(x) = x$. *Because* $(f \circ g)(x) = (g \circ f)(x) = x$, *then* f *and* g *are inverses of each other.*

We can sketch the graph of f^{-1} from the graph of f by using the following property:

Definition 5. <u>**Symmetry Property of the Graphs of f and f^{-1}**</u>**:** *Let* $f(x)$ *be a* ***one-to-one function****. Then the graph of* f *and the graph of* f^{-1} *are symmetric with respect to the line* $y = x$.

Definition 6. <u>**Finding f^{-1} of a One-to-One function:**</u> *Let* $f(x)$ *be a* ***one-to-one function****. Then to find* f^{-1} *we follow the following steps.*

1. *Replace $f(x)$ with y in the equation defining $f(x)$.*
2. *Interchange x and y.*
3. *Solve the equation in Step 2 for y.*
4. *Replace y with $f^{-1}(x)$.*

Example 4. *Find the inverse of the one-to-one function* $f(x) = \frac{x+1}{x-2}$, $x \neq 2$.
Solution:

Step 1. $y = \frac{x+1}{x-2}$ *Replace $f(x)$ with y.*

Step 2. $x = \frac{y+1}{y-2}$ *Interchange x and y.*

Step 3. *Solve* $x = \frac{y+1}{y-2}$ *for y.*

$$\begin{aligned} x &= \frac{y+1}{y-2} \\ x(y-2) &= y+1 \\ xy-2x &= y+1 \\ xy-2x+2x-y &= y-y+1+2x \\ xy-y &= 1+2x \\ y(x-1) &= 1+2x \\ y &= \frac{1+2x}{x-1} \end{aligned}$$

Step 4. $f^{-1}(x) = \frac{1+2x}{x-1}, \quad x \neq 1$ *Replace y with $f^{-1}(x)$.*